Regional Generalized Nonlinear Height-Diameter Model for Two Age Stands of Small-Leaved Linden

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Received:20.10.2025 Accepted:24.10.2025 Published:30.10.2025

https://doi.org/10.54414/OJCO2155

Abstract

Tree height and diameter at breast height (*dbh*) are essential for assessing quantitative stand characteristics, developing models of stand structure and productivity. The relationships between them (HD) represent a natural indicator of forest growth and serve as a measure of forest management practices. Measuring tree height is more challenging in uneven-aged forests compared to measuring diameters. To overcome these difficulties, the development of HD models is crucial. This study aimed to develop a reliable HD model for two-aged stands of *Tilia cordata* Mill. by incorporating stand-level variables. To effectively model the HD relationship, ten widely recognized nonlinear functions were evaluated. The selection of the best-performing model, which accurately describes the relationship between tree height and diameter, was based on the lowest root mean square error (*RMSE*), mean absolute percentage error (*MAPE*), and Akaike's information criterion (*AIC*), as well as the highest adjusted coefficient of determination (R^2-adj.), with statistically significant regression coefficients ($p < 0.05$). Performance statistics indicated that the three-parameter F.J. Richards function was the most suitable and is recommended for predicting HD relationships in small-leaved linden trees within the environmental region under study. This function was further modified by incorporating stand-level variables, such as mean height and quadratic mean diameter for each layer, as additional integrated predictors. The functions were fitted using nonlinear least squares. The fitted generalized HD model and its validation explained variability of 85–95% of the observed tree heights in small-leaved linden stands. This model enables the prediction of individual height curves for each layer within a given stand.

Keywords: small-leaved linden, two-aged stands, height-diameter model, nonlinear generalized model, validation, Richards function.

1. Introduction

Tree height and diameter are key parameters used to determine stem volume, estimate forest aboveground biomass, and assess overall forest condition and structure (Sharma et al., 2016; Ciceu et al., 2023; Tanovski et al., 2023). The relationship between them (HD) serves as a natural indicator of forest growth and a measure of forest management practices. Consequently, research on the height-diameter relationship holds significant scientific and practical importance, prompting numerous studies aimed at identifying mathematical models that most accurately describe this dependence (Ahmadi & Alavi, 2016; Dubenok et al., 2021).

A wide range of two- and three-parameter HD equations has been developed for various tree species across different regions (Seki, 2022). In these models, diameter serves as the predictor variable for estimating tree height. However, due to the structural heterogeneity of forest ecosystems, describing HD relationships using a single model is highly challenging (Sağlam & Sakici, 2024). In order to reduce variability in HD relationships and improve prediction accuracy, generalized models have been developed (Ismail et al., 2025). Generalized HD models incorporate not only diameter at breast height (DBH) but also individual tree- and stand-level variables. Literature reports such variables as age, site index, stand basal area, stem density, basal area of larger trees, dominant or mean height and diameter, among others (Sharma et al., 2016; Ahmadi & Alavi, 2016; Ciceu et al., 2020).

To our knowledge, no studies have been published on generalized height-diameter models for uneven-aged small-leaved linden (*Tilia cordata* Mill.) stands. Therefore, this research addresses a relevant gap by developing a generalized model for predicting tree height from DBH in two-aged linden stands. The study objectives were as follows: (1) selecting the most suitable model from a set of ten base models, (2) integrating stand-level variables that may influence the HD relationship to construct a generalized model, and (3) evaluating the predictive accuracy of the generalized model.

2. Material and Methods

The area under study, Arkhangelsk Forestry District, is situated in the central-eastern part of the Republic of Bashkortostan, the Russian Federation, along the western slope of the Southern Urals on the right bank of the Belaya River. The area is located between 54.21799–54.78734°N and 56.46937–57.43891°E, with elevations ranging from 150 to 900 m above sea level, classifying it as mountainous forest terrain. The climate is strongly continental, with an average annual precipitation of approximately 600 mm, more than half of which occurs during the growing season from May to September. Mean annual temperatures range from 2 to 6°C, and the frost-free period lasts 115 days. The predominant soils are gray forest soils.

Natural stands of small-leaved linden (*Tilia cordata* Mill.) in the study area cover approximately 95,000 ha with a growing stock exceeding 18 million m³. Mixed linden stands dominate, often including oak (*Quercus robur L.*), maple (*Acer platanoides L.*), elm (*Ulmus glabra Huds*. and *Ulmus laevis Pall*.), birch (*Betula pendula Roth*.), and aspen (*Populus tremula L.*), with occasional presence of spruce (*Picea obovata Ledeb*.), fir (*Abies sibirica Ledeb.*), and pine (*Pinus sylvestris L.*). Pure linden stands also occur. Uneven-aged linden stands account for about 18% of the total, most of which are two-aged, while three-aged stands are less common. The majority of uneven-aged linden stands are over 60 years old, with an age difference between generations (layers) ranging from 10 to 70 years.

Temporary sample plots (TSPs) ranging from 0.5 to 1.0 ha in size were used for the study: ten for modeling and three for testing. On each TSP, a complete inventory of tree diameters was conducted by layers, and heights of sample trees (randomly selected, 1-10 specimens per 1-cm diameter class) were measured. Diameter at breast height (*dbh*, 1.3 m) was measured using calipers, while tree height (*h*) was measured with a Blume-Leiss mechanical hypsometer with 0.5 m precision.

For each layer of the TSPs, three average-sized trees (60 stems total) were felled as sample trees. These trees were sectioned into 2-m segments, from which discs were obtained for complete growth analysis. The results of this growth analysis were subsequently used for mathematical modeling of the h-dbh relationship.

Based on the sample trees, stand age was determined to range from 70-95 years for the first layer and 50-75 years for the second layer. Some stand characteristics were reported earlier (Gabdelkhakov et al., 2025).

The training dataset comprised 2 842 tree measurements (first layer: *n*=1 435; second layer: *n*=1 407). For testing, h and *dbh* values from 391 trees were used (first layer: *n*=210; second layer: *n*=181).

Based on literature review, ten widely used equations (base models) for height-diameter relationships were initially selected and tested (Ismail et al., 2025; Tanovski et al., 2023). In the second stage, the selected base model was enhanced by integrating stand-level variables to develop a generalized model. Additional integrated stand variables included mean height (*H*) and quadratic mean diameter (*Dq*) for each layer. In the third stage, after fitting the generalized model to the training dataset, its predictive performance was evaluated using the test dataset.

Model fitting was performed using nonlinear least squares regression. The selection of the best model describing the height-diameter relationship was based on classical statistical metrics used in forest biometrics research. The model with the lowest root mean square error (*RMSE*), mean absolute percentage error (*MAPE*), and Akaike's information criterion (*AIC*), along with the highest adjusted coefficient of determination (R^2-adj.), was considered optimal (Ahmadi & Alavi, 2016). To visualize the magnitude and distribution of errors relative to DBH, relative error (RE) graph was developed (Tanovski et al., 2023). Additionally, the significance of model parameters was examined. Heteroscedasticity was assessed through graphical analysis of residual patterns and application of the Breusch-Pagan test (Breusch & Pagan 1979). All data analyses were conducted using StatSoft Statistica and Microsoft Excel.

3. Results and Discussion

Figure 1 presents scatter plots of diameter and height by stand layer based on growth analysis data from sample trees. The diagrams demonstrate significant correlations of r = 0.957 and r = 0.931 for the first and second layers, respectively. This relationship exhibits complex nonlinear behavior (Lebedev, 2020a), with the height curve increasing more rapidly during early growth stages than in later phases (Sharma et al., 2016). Notably, the height growth rate was higher in the supressed layer compared to the dominant layer.

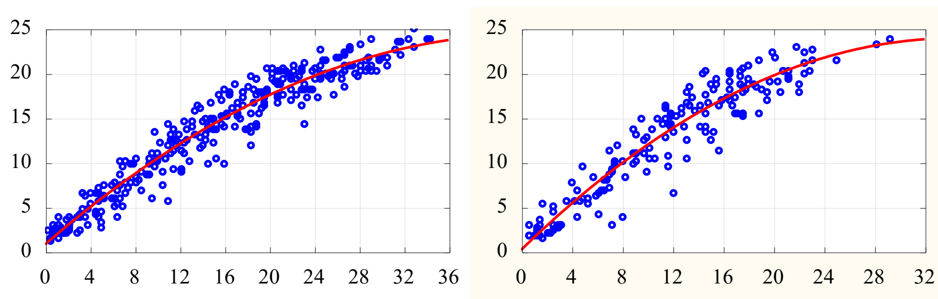

Figure 1. Scatter diagram of h (vertically, m) depending on dbh (horizontally, cm) for sampled trees in small-leaved linden stands (a – dominant layer, b – suppressed layer)

Analysis of the presented HD relationships reveals that models relying solely on DBH for height prediction in uneven-aged linden stands are insufficient for explaining tree height variation and fail to meet effective forest management requirements. Consequently, incorporating additional stand-level variables is necessary to improve height prediction accuracy.

Height-diameter functions must satisfy specific criteria. First, the equation's intercept must equal 1.3 m (Sharma & Parton, 2007; Ismail et al., 2025; Tanovski et al., 2023). Second, the curve must be monotonically increasing with a horizontal asymptote (Lebedev & Kuzmichev, 2020b; Tanovski et al., 2023). Regression analysis determined parameters for each of ten candidate models (not shown) in small-leaved linden stands. The multitude of growth-influencing factors results in substantial height variation among trees of equal DBH, significantly affecting model performance. Evaluation of ten equations identified the three-parameter F.J. Richards function (Richards, 1959) as demonstrating superior accuracy:

$$h = 1.3 + b_1(1 - \exp(-b_2 dbh))^{b_3} \quad (1)$$

where b_1, b_2, b_3 – model parameters.

This equation has been effectively used in nonlinear HD models by numerous researchers (Sharma & Parton, 2007; Ahmadi et al., 2013; Özçelik et al., 2018; Ciceu et al., 2020, 2023; Sağlam & Sakici, 2024), demonstrating high predictive performance. The generalized Richards model incorporating additional stand-level variables (mean height (H) and quadratic mean diameter (D_q) for each canopy layer) takes the following form (Lebedev, 2020a):

$$h = 1.3 + \frac{(H - 1.3)(1 - \exp(-(a_1 + a_2 D_q)^{(\frac{dbh}{D_q})}))^{a_3 + a_4 D_q}}{(1 - \exp(-(a_1 + a_2 D_q)))^{a_3 + a_4 D_q}} \qquad (2)$$

where a_1, a_2, a_3, a_4 – model parameters.

Table. Model estimates

Parameter	Estimate	*RMSE*	*MAPE*	R^2-adj	*AIC*
Base function (dominant canopy layer)					
b_1	22,15356	1,433	7,483	0,913	522
b_2	0,12712				
b_3	2,57598				
Base function (supressed canopy layer)					
b_1	18,11063	1,652	8,209	0,744	712
b_2	0,17306				
b_3	2,19962				
Generalized function (training data)					
a_1	3,28804	1,086	3,992	0,941	244
a_2	-0,00994				
a_3	3,27137				
a_4	-0,07211				
Generalized function (validation data)					
a_1	3,28804	1,067	4,175	0,857	33
a_2	-0,00994				
a_3	3,27137				
a_4	-0,07211				

The final statistical indicators for the base model (1) of both layers and the generalized model (2) are presented in the table. The quality metrics (*RMSE*, *MAPE*, R^2-adj., and *AIC*) demonstrate strong predictive capability of the selected model for both layers. The generalized model, incorporating additional stand variables of the mean height H and quadratic mean diameter Dq of each layer, exhibited excellent fit, explaining over 94% of observed variability (R^2-adj.) with $RMSE < 1.1$ m, $MAPE < 4.0\%$, and low *AIC* values. The generalized model showed minimal deviations between predicted and measured heights. Model parameters $a_{1\text{-}4}$ had low standard errors (0.418, 0.003, 0.204, and 0.007, respectively), with statistically significant estimates of p-value < 0.001. The model's high predictive accuracy partly stems from including data from average sample trees, though we assumed these trees remained representative of their layers throughout their lifespan (an idealization that may not always hold).

Nonlinear models used in forestry for height prediction often exhibit heteroscedasticity, where error variance is non-constant across observations. Visual residual analysis and Breusch-Pagan tests confirmed the absence of heteroscedasticity. Residual graphs (not shown) supported assumptions of normality, homoscedasticity, and independence — residuals were randomly distributed without discernible patterns. These findings align with prior studies (Sharma & Parton, 2007; Siipilehto et al., 2023; Ismail et al., 2025), where the asymptotic Richards model performed well in its original form and improved further after modification.

Application to test data yielded consistent results. Minor differences in performance metrics between training and testing phases arose from inherent dataset variations. Figure 2 shows that relative errors (RE) for data were generally small, though slight overestimation occurred for both thinner and larger trees. The three-parameter base model, when generalized, provided greater flexibility for HD curves, corroborating other research (Lebedev, 2020a; Siipilehto et al., 2023; Ismail et al., 2025).

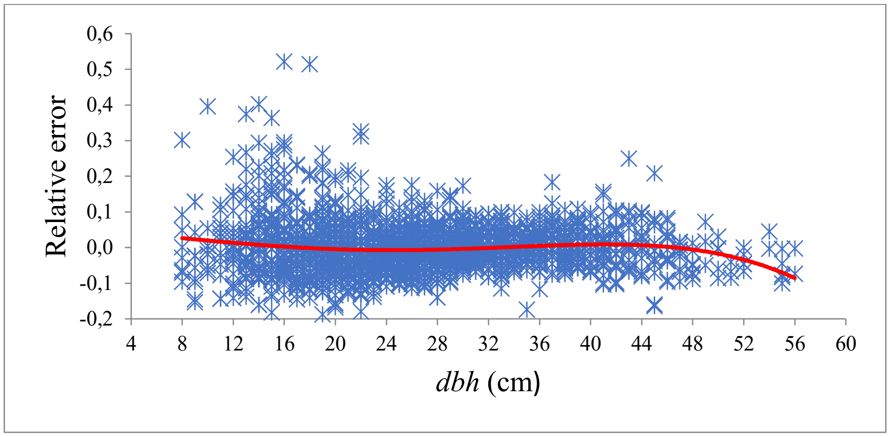

Figure 2. Relative error

4. Conclusions

This study evaluated several base HD models to develop an optimal height-diameter relationship model for two-aged small-leaved linden (*Tilia cordata* Mill.) stands. The three-parameter Richards model, grounded in growth theory, demonstrated superior performance by accurately capturing accelerated early-stage height growth followed by slower maturation rates, while also accounting for the greater growth velocity of the supressed canopy layer. The fitted generalized HD model and its validation explained the variability of 85–94% of the observed height in small-leaved linden stands across the area under study.

The model enables stand-specific height curve predictions for individual canopy strata using three readily obtainable variables: diameter at breast height (DBH), quadratic mean diameter, and mean stand height. The quadratic mean diameter serves as a proxy for stand density and competition intensity, exhibiting strong correlations with stems per hectare, while mean height complements this by accounting for height variability. These variables are routinely collected in forest inventories and can be projected forward using growth equations. The developed equation can be used for forest inventories and as primary input data for growth and yield models when formulating forest management plans.

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